On the absence of dissipative instability of negative energy waves in hydrodynamic shear flows.

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Stability criterion for the *surface gravity capillary waves* in a flowing two-layered fluid system with viscous dissipation is investigated. It is seen that the *dissipative instability* of negative energy waves is absent,- contrary to what earlier authors have concluded. Their error is identified to arise from an erroneous choice of the dissipation law, in which the wave profile velocity is wrongly equated to the particle velocity. Our corrected dissipation law is also shown to restore *Galilean invariance* to the stability condition of the system.

Dissipative instability in a flowing system has been discussed in the literature, and is being sought to be applied to explain various phenomena for a few decades. This instability was shown by many authors to be caused by viscosity, but seen only in some selected frames of reference in which a particular mode of the system possesses *negetaive energy* [1–11]. Their result, therefore, implies that the stability of such a system is frame dependent, which obviously violates the condition of *Galilean invariance.* In the present paper, it is shown that the violation of Galilean invariance arose due to a wrong choice of the dissipation law. This question has also been recently addressed by us in an earlier paper [12], where we calculated the total energy in a *magnetohydrodynamic shear flow.* It is there shown that the existing theories give a rate of entropy production which is not invariant under Galilean transformation. The method to calculate the correct rate of energy dissipation is given by us in the earlier paper in considerable detail. In this research note, we analyse the problem from the point of view of Euler's equation of motion and show the validity of our earlier conclusions.

The system considered is a single interface of discontinuity separating two uniform *incompressible* fluid media. The interface is along the x- direction with the force of gravity acting downwards, i.e., in the negative z- direction. The equilibrium pressures, densities and velocities are given by

$$p_0(z), \ \rho_0(z), \ u_0(z) = \begin{cases} p_1, \ \rho_1, \ u_1, \ z \leq 0, \\ p_2, \ \rho_2, \ u_2, \ z > 0, \end{cases} \tag{1}$$

with the equlibrium pressure balance condition requiring that $p_1 = p_2$.

We consider small perturbations about the above equilibrium configuration. The vertical displacement $\eta(x, z, t)$ of a streamline at any point (x, z) can then be written in a form

$$\eta(x, z, t) \equiv \begin{cases} \tilde{\eta}(k, t) \exp(ikx) \exp\{kz\}, & z \leq 0, \\ \tilde{\eta}(k, t) \exp(ikx) \exp\{-kz\}, & z > 0, \end{cases} \tag{2}$$

that is consistent with an incompressible and irrotational flow. From the *linearised equations of hydrodynamics*, we then obtain the expressions for the velocity and the pressure fluctuations. These expressions are (with $i =$1,2)

$$\delta u_{xi} = \mp (n + ku_i) \, \tilde{\eta}(k, t) \exp(ikx) \exp\{\pm(kz)\}, \tag{3 a}$$

$$\delta u_{zi} = \left(\frac{n + ku_i}{n} \right) \dot{\tilde{\eta}}(k, t) \exp(ikx) \exp\{\pm(kz)\}, \tag{3 b}$$

and

$$\delta p_i = \pm \left(\frac{\rho_i}{k} \right) (n + ku_i)^2 \, \tilde{\eta}(k, t) \exp(ikx) \exp\{\pm(kz)\}, \tag{3 c}$$

with the upper sign designating $i = 1$ and the lower sign designating $i = 2$. In Eq.(3), we have assumed the temporal dependence to be $\tilde{\eta}(k, t) \sim \exp(int)$, with a 'dot' designating a time derivative $\partial/\partial t$.

At the interface ($z = 0$), the continuity of pressure fluctuations demand

$$\delta p_1(x, t) - \delta p_2(x, t) = -T \frac{\partial^2}{\partial x^2} \eta(x, t) + g (\rho_1 - \rho_2) \eta(x, t), \tag{4}$$

with g being the acceleration due to gravity, and T being the suface tension of the interface. While solving for the temporal Fourier amplitudes (i.e., the normal modes) $\overline{\eta}(k,n)$ by substituing (3) in (4), we find that for a non-trivial solution to exist, one must have

$$\left(n + k\overline{U}\right)^2 = (\Delta n)^2, \tag{5}$$

where,

$$(\Delta n)^2 = gk\left(\alpha_1 - \alpha_2\right) + k^3 T' - k^2 \alpha_1 \alpha_2 \left(u_1 - u_2\right)^2, \tag{6}$$

with

$$\alpha_i = \rho_i / \left(\rho_1 + \rho_2\right), (i = 1, 2), \tag{7 a}$$

$$\overline{U} = \left(\alpha_1 u_1 + \alpha_2 u_2\right), \tag{7 b}$$

and

$$T' = T / \left(\rho_1 + \rho_2\right). \tag{7 c}$$

Eqs.(5-7) constitute the familiar dispersion relation for the *surface gravity capillary waves* that exihibits the *Rayleigh-Taylor* and the *Kelvin-Helmholtz instabilities* under certain conditions [13].

It is necessary at this stage to introduce the question of dissipative instability. It was argued by several authors [1–11] that, on introducing a kinematic viscosity ν in any one of the two media (in the lower medium, say), one changes the dispersion relation to

$$\left(n + k\overline{U}\right)^2 = (\Delta n)^2 + i\nu\alpha_1 nk^2, \tag{8}$$

which, for $(\Delta n)^2 > 0$ and for a small kinematic viscosity $\left(\nu\alpha_1 k^3 \overline{U}/(\Delta n)^2 \ll 1\right)$, gives the two roots as

$$n_\pm = -k\left(\overline{U} \pm (\Delta n)/k\right) + \frac{i\nu\alpha_1 k^2}{2(\Delta n)}\left[(\Delta n) \pm k\overline{U}\right]. \tag{9}$$

For $\overline{U} > (\Delta n)/k$, the (-) root in Eq.(9) is called a *negative energy wave* that grows in time as $\exp\left[\nu\alpha_1 k^2 \left(k\overline{U}/(\Delta n) - 1\right)/2\right]$, thus giving *dissipative instability* of the negative energy wave. Note that, the growth is possible for any non-zero but small value of ν whenever $\overline{U} > (\Delta n)/k$, while for ν exactly equal to zero, the instability criterion has no dependence on $\overline{U}$ and is given by [13] $(\Delta n)^2 < 0$, or,

$$k^2 \alpha_1 \alpha_2 \left(u_1 - u_2\right)^2 > gk\left\{\left(\alpha_1 - \alpha_2\right) + k^2 T'/g\right\}. \tag{10}$$

The above result contains two surprising conclusions. Firstly, the response of the system is not continuous with respect to ν as $\nu \longrightarrow 0$, i.e., the stability of the system for an arbitrarily small viscosity is different from that when viscosity is exactly zero. Furthermore, the stability with a small but nonzero ν appears to be dependent on $\overline{U}$, where $\overline{U}$ is of course dependent on the frame of reference. This means, that the stability of the system depends on the frame of reference of the observer. In other words, by moving the observer with a given speed, one can create an instability of the negative energy waves,- a result which obviously violates the fundamental law of *Galilean invariance*.

The problem can, however, be resolved in the following way. As has been done by most authors, we consider one of the fluids, i.e., the lower one to be viscous, while the upper one to be non-viscous. This simplification enables us to ignore the complications due to boundary layers, while the essential physics remains unaltered. We note, that by substituting $in \equiv \partial/\partial t$ and $ik \equiv \partial/\partial x$ in the dispersion relation (5) for the non-viscous case, we obtain an equation of motion

$$\frac{D^2}{Dt^2}\eta(x,t) = -(\Delta n)^2 \eta(x,t), \tag{11}$$

where, D/Dt is the total derivative

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x}. \tag{12}$$

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From Eq.(12), it is seen that, Eq.(11) describes the force equation of a fluid system moving with a nett velocity $\overline{U}$ with respect to the given frame of reference. Due cognizance should be taken about this fact while calculating the viscous force term $\nu\rho_1\partial^2/\partial x^2(\delta v_z)$. It is to be noted that the term δv_z here denotes the *real velocity of the fluid particles* pertaining to wave propagation [14], and not the profile velocity $\partial\eta(x,t)/\partial t$, as the earlier authors have suggested. We note that, in this moving fluid system, the particle velocity is calculated as $\delta v_z = D\eta(x,t)/Dt = i(n+k\overline{U})\eta(x,t)$.

The above argument implies that, the equation of motion in the presence of viscosity must read

$$\frac{D^2}{Dt^2}\tilde{\eta}(k,t) = -(\Delta n)^2\tilde{\eta}(k,t) - \nu\alpha_1 k^2\frac{D}{Dt}\tilde{\eta}(k,t), \tag{13}$$

thus giving a dispersion relation

$$\left(n+k\overline{U}\right)^2 = (\Delta n)^2 + i\nu\alpha_1\left(n+k\overline{U}\right)k^2, \tag{14}$$

that yields the two roots

$$n_\pm = -k\left[\overline{U}\pm\frac{1}{k}\left\{(\Delta n)^2 - \nu^2\alpha_1^2 k^4/4\right\}^{1/2}\right] + i\nu\alpha_1 k^2/2. \tag{15}$$

In Eq.(15), the last term on the right hand side predicts a damping for both the wavemodes when $(\Delta n)^2 > \nu^2\alpha_1^2 k^4/4$. A growth is, however, possible if and only if

$$(\Delta n)^2 - \nu^2\alpha_1^2 k^4/4 < 0, \text{ and } \left\{\nu^2\alpha_1^2 k^4/4 - (\Delta n)^2\right\}^{1/2} > \nu\alpha_1 k^2/2, \tag{16}$$

thus presenting the same instability criterion $(\Delta n)^2 < 0$, as in equation (10) for the non-viscous case. While precluding the dissipative instability of negative energy waves, Eqs.(15) and (16) thus suggest that the presence of viscous dissipation does not at all alter the stability property of the surface gravity capillary waves.

The foregoing analysis shows that, the dissipative instability is simply an artifact of an erroneous choice of the viscous damping law by the earlier authors. The correct viscosity law, written as in the last term of Eq.(13), corresponds to a resistance proportional to $\nu\rho_1\partial^2(\delta v_z)/\partial x^2 = -i\nu\rho_1 k^2\left(n+k\overline{U}\right)\eta(x,t)$. Replacing this expression by $-i\nu\rho_1 nk^2\eta(x,t)$, as was done by the earlier authors, would be equivalent to having a resistance of the form $\alpha\left(n,\overline{U}\right)\nu\rho_1\partial^2(\delta v_z)/\partial x^2$, with $\alpha(n,\overline{U}) = n/\left(n+k\overline{U}\right)$. For small values of the kinematic viscosity ν, we can then use Eq.(5) to write $\alpha(n_+,\overline{U}) \approx \left(k\overline{U}+\Delta n\right)/\Delta n$ and $\alpha(n_-,\overline{U}) \approx \left(-k\overline{U}+\Delta n\right)/\Delta n$ for the (+) and the (-) modes, respectively. Here, $\alpha(n_+,\overline{U})$ is always positive, but $\alpha(n_-,\overline{U})$ is negative when $\overline{U} > (\Delta n)/k$. In such a situation, the force $\alpha(n_-,\overline{U})\nu\rho_1\partial^2(\delta v_z)/\partial x^2$ would act as an attractive force, that helps to build up the amplitude of the negative energy wave. The erroneous resistance formula, that has been used so far in the literature, thus makes the viscous drag force frame dependent and gives a velocity dependent acceleration in selected frames, rather than a deceleration in all frames. By substituting unity for $\alpha(n,\overline{U})$, the correct viscous resistance formula however demands that, the velocity δv_z be the true velocity of the fluid particles pertaing to wave motion so that, the viscous force becomes frame independent. The use of this correct formula leads us to two important results. Firstly, it precludes the possibility of dissipative instability in negative energy wave systems. Secondly, it gives a dispersion relation in which the drift velocity $\overline{U}$ appears only as a Doppler shift term in the frequency;- the important consequence of which is that, the stability of the system is independent of $\overline{U}$, thus giving the required frame independent universal character to the stability condition of the system.

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